Discrete Mathematics 309 (2009) 3150–3155

Contents lists available at ScienceDirect

Discrete Mathematics

journal homepage: www.elsevier.com/locate/disc

A classification of cubic s-regular graphs of order $16p$

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ARTICLE INFO

Article history:
Received 25 November 2007
Received in revised form 10 August 2008
Accepted 1 September 2008
Available online 21 September 2008

Keywords:
Cubic s-regular graph
Covering graph
The Möbius–Kantor graph

ABSTRACT

A graph is *s-regular* if its automorphism group acts regularly on the set of its s-arcs. In this paper, we classify all cubic s-regular graphs of order $16p$ for every $s \geq 1$ and every prime p. As a result, a new infinite family of cubic 1-regular graphs with girth 10 is constructed.

1. Introduction

Throughout this paper, graphs are finite, simple, undirected and connected. For a graph X, let $V(X)$, $E(X)$ and $\mathrm{Aut}(X)$ denote the vertex set, the edge set and the full automorphism group of X, respectively. The *arc set* $A(X)$ of a graph X is defined to be the set $\{(u, v), (v, u) \mid \{u, v\} \in E(X)\}$. For a vertex $v \in V(X)$, by $N(v)$ we denote the set of vertices adjacent to v. A graph $\widetilde{X}$ is called a *covering* of X with a projection $p : \widetilde{X} \to X$ if p is a surjection from $V(\widetilde{X})$ to $V(x)$ such that $\mathbf{p}|_{N(\widetilde{v})} : N(\widetilde{v}) \to N(v)$ is a bijection for any vertex $v \in V(X)$ and $\widetilde{v} \in \mathbf{p}^{-1}(v)$. The graph X is usually referred to as the *base graph* and $\widetilde{X}$ as the *covering graph*. The *fibre* of an arc or a vertex is its preimage under p. The group $\mathrm{CT}(\mathbf{p})$ of all automorphisms of $\widetilde{X}$ which fix each of the fibres setwise is called the *covering transformation group*.

An *s-arc* in a graph X is an ordered $(s + 1)$-tuple $(v_0, v_1, \ldots, v_s)$ of vertices of X such that v_{i-1} is adjacent to v_i for $1 \leq i \leq s$, and $v_{i-1} \neq v_{i+1}$ for $1 \leq i < s$. A graph X is said to be *s-arc-transitive* if $\mathrm{Aut}(X)$ is transitive on the set of s-arcs in X. In particular, 0-arc-transitive means *vertex-transitive*, and 1-arc-transitive means *arc-transitive* or *symmetric*. A subgroup of the automorphism group of a graph X is said to be *s-regular* if it acts regularly on the set of s-arcs of X. In particular, if the subgroup is the full automorphism group $\mathrm{Aut}(X)$ of X then X is said to be s-regular. Thus, if a graph X is s-regular then $\mathrm{Aut}(X)$ is transitive on the set of s-arcs and the only automorphism fixing an s-arc is the identity automorphism of X.

A covering $\mathbf{p} : \widetilde{X} \to X$ is said to be *regular* (or *N-covering*) if there is a semiregular subgroup N of the automorphism group $\mathrm{Aut}(\widetilde{X})$ such that the graph X is isomorphic to the quotient graph $\widetilde{X}/N$, say by an isomorphism h, and the quotient map $\widetilde{X} \to \widetilde{X}/N$ is the composition $h\mathbf{p}$ of h and $\mathbf{p}$. If the covering graph $\widetilde{X}$ is connected, then N is the covering transformation group. An automorphism of a covering graph $\widetilde{X}$ is said to be *fibre-preserving* if it maps a fibre to a fibre, while a covering transformation maps a fibre onto itself. An automorphism $\alpha \in \mathrm{Aut}(X)$ *lifts along* $\mathbf{p}$ if there exists an automorphism $\widetilde{\alpha} \in \mathrm{Aut}(\widetilde{X})$ such that $\alpha\mathbf{p} = \mathbf{p}\widetilde{\alpha}$. In this case we also say that $\mathbf{p}$ is *α-admissible*. A subgroup $G \leq \mathrm{Aut}(X)$ *lifts along* $\mathbf{p}$ if each $\alpha \in G$ lifts. The set of all lifts G forms a group $\widetilde{G} \leq \mathrm{Aut}(\widetilde{X})$, called the *lift* of G. A regular covering projection $\mathbf{p}$ is *arc-transitive* if some arc-transitive subgroup of $\mathrm{Aut}(X)$ lifts along $\mathbf{p}$.

Two coverings $\widetilde{X}$ and $\widetilde{X}'$ with projections $\mathbf{p}$ and $\mathbf{p}'$, respectively, are said to be *isomorphic* if there exist an automorphism $\alpha \in \mathrm{Aut}(X)$ and an isomorphism $\widetilde{\alpha} : \widetilde{X} \to \widetilde{X}'$ such that $\alpha\mathbf{p} = \mathbf{p}'\widetilde{\alpha}$. In particular, if α is the identity automorphism of X, then we say that $\widetilde{X}$ and $\widetilde{X}'$ are *equivalent*.

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0012-365X/$ – see front matter
doi:10.1016/j.disc.2008.09.001

Table 1
Cubic symmetric graphs of order $16p$ with $p \leq 47$

Graph	Order	s-regular	Girth	Diameter	Bipartite?
F_{32}	$16 \cdot 2$	2	6	5	Yes
F_{48}	$16 \cdot 3$	2	8	6	Yes
F_{80}	$16 \cdot 5$	3	10	8	Yes
F_{112A}	$16 \cdot 7$	1	10	7	Yes
F_{112B}	$16 \cdot 7$	2	8	7	Yes
F_{112C}	$16 \cdot 7$	3	8	10	Yes
F_{208}	$16 \cdot 13$	1	10	9	Yes
F_{304}	$16 \cdot 19$	1	10	11	Yes
F_{496}	$16 \cdot 31$	1	10	15	Yes
F_{592}	$16 \cdot 37$	1	10	15	Yes
F_{688}	$16 \cdot 43$	1	10	17	Yes

Let X be a connected graph and N be a finite group, called the *voltage group*. Assign to each arc of X a *voltage* $\xi(u, v) \in N$ such that $\xi(v, u) = \xi(u, v)^{-1}$. This function ξ is called an (*ordinary*) *voltage assignment* of X. Let $\mathrm{Cov}(X, \xi)$ be the *derived graph* with vertex set $V \times N$ and adjacency relation defined by $(u, a) \sim (v, a\xi(u, v))$ whenever $u \sim v$ in X. Then the first coordinate projection is a regular covering $\mathbf{p}_\xi : \mathrm{Cov}(X, \xi) \to X$ where the group N, viewed as $\mathrm{CT}(\mathbf{p}_\xi)$, acts via left multiplication on itself. Given a spanning tree T of the graph X, a voltage assignment ξ is called T-*reduced* if the voltages on the tree arcs are the identity. Gross and Tucker [14] showed that every regular covering projection $\mathbf{p} : \widetilde{X} \to X$ is equivalent to $\mathbf{p}_\xi : \mathrm{Cov}(X, \xi) \to X$ for some T-reduced voltage assignment $\xi : X \to N$ with respect to an arbitrary fixed spanning tree T of X.

Tutte [25,26] showed that every finite cubic symmetric graph is s-regular for some $s \geq 1$, and this s is at most five. It follows that every cubic symmetric graph has an order of the form $2mp$ for a positive integer m and a prime number p. In order to know all cubic symmetric graphs, we need to classify the cubic s-regular graphs of order $2mp$ for a fixed positive integer m and each prime p. Conder and Dobcsányi [3,4] classified the cubic s-regular graphs up to order 2048 with the help of the "Low index normal subgroups" routine in MAGMA system [1]. Cheng and Oxley [2] classified the cubic s-regular graphs of order $2p$. Recently, by using the covering technique, cubic s-regular graphs with order $2p^2$, $2p^3$, $4p$, $4p^2$, $6p$, $6p^2$, $8p$, $8p^2$, $10p$, $10p^2$ and $14p$ were classified in [7–12,21].

In this paper, we classify all cubic s-regular cubic graphs with order $16p$ for each $s \geq 1$ and each prime p. As a result, a new infinite family of cubic 1-regular graphs with girth 10 is constructed.

2. The cubic symmetric graphs of order 16*p*

We will use the following well-known results in group theory.

Proposition 2.1 ([15, Chapter IV, Theorem 2.6]). *Let G be a finite group and P a Sylow p-subgroup of G. Let $N_G(P)$ be the normalizer of P in G and $C_G(P)$ the centralizer of P in G. If $N_G(P) = C_G(P)$, then G has a normal subgroup N such that $G/N \cong P$.*

Proposition 2.2. (1) [22, *Theorem 8.5.3*] *Let p and q be primes and let a and b be non-negative integers. Then every group of order $p^a q^b$ is solvable.*
(2) [13, *Feit–Thompson Theorem*] *Every finite group of odd order is solvable.*

Let X be a graph and let N be a subgroup of $\mathrm{Aut}(X)$. Denote by $\underline{X}$ the quotient graph corresponding to the orbits of N, that is the graph having the orbits of N as vertices with two orbits adjacent in $\underline{X}$ whenever there is an edge between those orbits in X.

Proposition 2.3 ([16, Theorem 9]). *Let X be a connected symmetric graph of a prime valency and let G be an s-arc-transitive subgroup of $\mathrm{Aut}(X)$ for some $s \geq 1$. If a normal subgroup N of G has more than two orbits, then it is semiregular and G/N is an s-arc-transitive subgroup of $\mathrm{Aut}(\underline{X})$ where $\underline{X}$ is the quotient graph of X corresponding to the orbits of N.*

By [3,4] we have the following.

Lemma 2.4. *Let p be a prime. Let X be a cubic symmetric graph of order $16p$. If $p \leq 47$, then X is isomorphic to one of the graphs in Table 1.*

Assume that a connected graph X and a subgroup $G \leq \mathrm{Aut}(X)$ are given. Choose a spanning tree T of X and a set of arcs $\{x_1, \ldots, x_r\} \subseteq A(X)$ containing exactly one arc from each edge in $E(X \setminus T)$. Let $\mathcal{B}_T$ be the corresponding basis of the first homology group $H_1(X, \mathbb{Z}_p)$ determined by $\{x_1, \ldots, x_r\}$. Further, denote by $G^{\#_h} = \{\alpha^{\#_h} \mid \alpha \in G\} \leq \mathrm{GL}(H_1(X, \mathbb{Z}_p))$ the induced action of G on $H_1(X, \mathbb{Z}_p)$, and let $M_G \leq \mathbb{Z}_p^{r \times r}$ be the matrix representation of $G^{\#_h}$ with respect to the basis $\mathcal{B}_T$. By M_G^t we denote the dual group consisting of all transposes of matrices in M_G.

The following proposition is a special case of [18, Proposition 6.3, Corollary 6.5] (also see [6,23]).

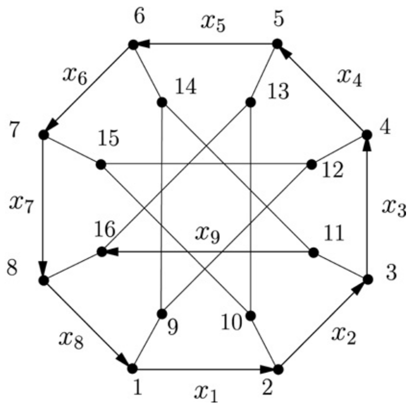

Fig. 1. The Möbius–Kantor graph.

Proposition 2.5. *Let T be a spanning tree of a connected graph X and let the set $\{x_1, x_2, \ldots, x_r\} \subseteq A(X)$ contain exactly one arc from each cotree edge. Let $\xi : A(X) \to \mathbb{Z}_p$ be a voltage assignment on X which is trivial on T, and let $Z(\xi) = [\xi(x_1), \xi(x_2), \ldots, \xi(x_r)]^t \in \mathbb{Z}_p^{r\times 1}$. Then the following holds.*

(a) *A group $G \leq \mathrm{Aut}(X)$ lifts along $p_\xi : \mathrm{Cov}(X, \xi) \to X$ if and only if the induced subspace $\langle Z(\xi)\rangle$ is an M_G^t-invariant 1-dimensional subspace.*
(b) *If $\xi' : A(X) \to \mathbb{Z}_p$ is another voltage assignment satisfying (a), then $\mathrm{Cov}(X, \xi')$ is equivalent to $\mathrm{Cov}(X, \xi)$ if and only if $\langle Z(\xi)\rangle = \langle Z(\xi')\rangle$, as subspaces. Moreover, $\mathrm{Cov}(X, \xi')$ is isomorphic to $\mathrm{Cov}(X, \xi)$ if and only if there exists an automorphism $\alpha \in \mathrm{Aut}(X)$ such that the matrix M_α^t maps $\langle Z(\xi')\rangle$ onto $\langle Z(\xi)\rangle$.*

The Möbius–Kantor graph F_{16} is illustrated in Fig. 1. It is known that F_{16} is a unique cubic symmetric graph of order 16, which is 2-regular (see [3,4]). We choose

$$\alpha := (2, 8, 9)(3, 16, 14)(4, 13, 6)(7, 12, 10),$$
$$\beta := (1, 2)(3, 8)(4, 7)(5, 6)(9, 10)(11, 16)(12, 15)(13, 14),$$
$$\gamma := (1, 2)(3, 9)(4, 14)(5, 6)(7, 13)(8, 10)(11, 12)(15, 16)$$

as automorphisms of F_{16}. Then $\mathrm{Aut}(F_{16}) = \langle \alpha, \beta, \gamma\rangle$ and $\mathrm{Aut}(F_{16})$ has two proper arc-transitive subgroups $H := \langle \alpha, \beta\rangle$ and $K := \langle \alpha, \gamma\rangle$. This can be checked by GAP [24].

Thus, in order to determine all arc-transitive $\mathbb{Z}_p$-covering projections of F_{16}, it suffices to find those which are H- or K-admissible. By Proposition 2.5, this is equivalent to finding all invariant 1-dimensional subspaces of the representations M_H^t or M_K^t.

We choose a spanning tree T of F_{16} consisting of the edges

$$\{\{1, 9\}, \{2, 10\}, \{3, 11\}, \{4, 12\}, \{5, 13\}, \{6, 14\}, \{7, 15\}, \{8, 16\},$$
$$\{9, 12\}, \{9, 14\}, \{10, 13\}, \{10, 15\}, \{11, 14\}, \{12, 15\}, \{13, 16\}\}.$$

We orient the cotree arcs by setting

$$x_1 = (1, 2), \quad x_2 = (2, 3), \quad x_3 = (3, 4), \quad x_4 = (4, 5), \quad x_5 = (5, 6),$$
$$x_6 = (6, 7), \quad x_7 = (7, 8), \quad x_8 = (8, 1), \quad x_9 = (11, 16).$$

Let $\mathcal{B} = \{C_{x_i} \mid 1 \leq i \leq 9\}$ be the standard ordered basis of $H_1(F_{16}, \mathbb{Z}_p)$ associated with the spanning tree T and the arcs x_i $(i = 1, \ldots, 9)$. Let $p \geq 5$ be a prime number such that $p = 1 \bmod (6)$ and let ζ be a solution of the equation $x^2 + x + 1 = 0$ in $\mathbb{Z}_p$. We define a T-reduced voltage assignment $\xi : \{x_i \mid 1 \leq i \leq 9\} \to \mathbb{Z}_p$ by setting

$$x_1 \mapsto \zeta, \quad x_2 \mapsto 1 - \zeta, \quad x_3 \mapsto \zeta, \quad x_4 \mapsto -\zeta - 1, \quad x_5 \mapsto \zeta + 2,$$
$$x_6 \mapsto -\zeta - 1, \quad x_7 \mapsto \zeta, \quad x_8 \mapsto 1 - \zeta, \quad x_9 \mapsto -2.$$

We remark that the voltage assignment ξ is derived from the M_H^t-invariant 1-dimensional subspace $\langle k_1\rangle$ (see Section 3). Let $CF_{16p}(p \geq 5)$ be the derived graph from the voltage assignment ξ.

Malnič et al. [17] classified *semisymmetric elementary abelian* covers of F_{16}. One might derive the following theorem from [17]. But, we give its (simpler) proof in the next section.

Theorem 2.6. *Let $p \geq 5$ be a prime. Let $\widetilde{X}$ be an arc-transitive $\mathbb{Z}_p$-cover of the Möbius–Kantor graph F_{16}. Then $\widetilde{X}$ is isomorphic to the 1-regular graph CF_{16p} of girth 10 where $p = 1 \bmod (6)$.*

Remark. Marušič et al. [19,20] gave the relation between *half-transitive* group action with vertex stabilizer $\mathbb{Z}_2$ and 1-regular group action with cyclic vertex stabilizer, which give us infinitely many finite half-transitive graphs of valency 4.

The following is the main result in this paper.

Theorem 2.7. *Let p be a prime and let X be a connected cubic symmetric graph of order $16p$. Then X is 1-, 2- or 3-regular. Furthermore,*

(1) *X is 1-regular if and only if X is isomorphic to the graph $CF_{16p}(p \geq 7)$, where $p = 1 \bmod (6)$.*
(2) *X is 2-regular if and only if X is isomorphic to one of the three graphs F_{32}, F_{48} and F_{112B}.*
(3) *X is 3-regular if and only if X is isomorphic to one of the two graphs F_{80} and F_{112C}.*

Proof. Let X be a cubic symmetric graph of order $16p$. By [3,4] we may assume $p > 47$. Let $A = \mathrm{Aut}(X)$ and let P be a Sylow p-subgroup of A. If P is normal in A, by Proposition 2.3 X is a regular covering of the graph F_{16} with the covering transformation group $\mathbb{Z}_p$ and the normality of P implies that the fibre-preserving group is arc-transitive. By Theorem 2.6, X is isomorphic to CF_{16p}. Thus, it suffices to show that P is normal in A.

Let $N_A(P)$ be the normalizer of P in A. By Sylow's theorem, the number of Sylow p-subgroups of A is $np + 1 = |A : N_A(P)|$. Since X is at most 5-regular, $|A|$ is a divisor of $48 \cdot 16p$. Thus $np + 1$ is a divisor of $48 \cdot 16$. Suppose to the contrary that P is not normal in A. Since $np + 1 \geq 54$ and $np + 1 \mid 2^8 \cdot 3$, we have $(n, p) = (13, 59), (1, 127), (1, 191)$ or $(1, 383)$. If $N_A(P) = P$ then $C_A(P) = P$, where $C_A(P)$ is the centralizer of P in A. By Proposition 2.1, A has a normal subgroup N such that $A/N \cong P$, and by Proposition 2.3, the quotient graph corresponding to the orbits of N has odd order and valency 3, a contradiction. Thus one may assume $(n, p) \neq (13, 59)$. Since $|A : N_A(P)| = 2^7, 2^6 \cdot 3$ or $2^7 \cdot 3$, $|A|$ has a divisor $2^7 \cdot 3 \cdot p$ where $p = 127, 191$ or 383, implying that X is at least 3-arc-transitive. Let M be a minimal normal subgroup of A and $\underline{X}$ the quotient graph of X corresponding to the orbits of M.

If M is elementary abelian then by Proposition 2.3 $\underline{X}$ is 3-arc-transitive with order 2^4, $2p$, $4p$ or $8p$, which is impossible by the result in [3,4], [8, Theorem 5.2] and [11, Theorem 5.1]. Thus, one may assume that $M = T_1 \times T_2 \times \cdots \times T_t$, where T_i $(1 \leq i \leq t)$ are isomorphic non-abelian simple groups. By Proposition 2.2, $|T_i|$ has at least three prime factors. Notice that $|A|$ is a divisor of $2^8 \cdot 3 \cdot p$ where $p = 127, 191$ or 383. Then $t = 1$ and M is a non-abelian simple group. Thus M has order $2^\ell \cdot 3 \cdot p$ for some $1 \leq \ell \leq 8$. However, there is no simple group with such orders (see [5]). $\square$

3. The proof of Theorem 2.6

Let $p \geq 5$ be a prime. It is known that a polynomial $x^2 + x + 1 = 0$ has a solution in $\mathbb{Z}_p$ if and only if -3 is a square root in $\mathbb{Z}_p$, which is if and only if $p = 1 (\bmod 6)$.

Let R, T and S be the *transposes* of the matrices which represent the linear transformations $\alpha^{\#_h}$, $\beta^{\#_h}$ and $\gamma^{\#_h}$ relative to $\mathcal{B}$, respectively. Then

$$R = \begin{bmatrix} -1 & 0 & 0 & 0 & 0 & 0 & -1 & -1 & 0 \\ 0 & -1 & 0 & 0 & 0 & 0 & 1 & 0 & -1 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 & 1 & 0 & 0 & 0 \\ 0 & -1 & -1 & -1 & -1 & -1 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \end{bmatrix},$$

$$T = \begin{bmatrix} -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 \end{bmatrix},$$

$$S = \begin{bmatrix} -1 & -1 & 0 & 0 & 0 & 0 & 0 & -1 & -1 \\ 0 & 0 & -1 & 0 & 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 1 & 0 & -1 \\ 0 & 0 & -1 & -1 & -1 & -1 & -1 & 0 & 1 \\ 0 & 0 & 1 & 1 & 0 & 0 & 0 & 0 & -1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & -1 & 0 & 1 \\ 0 & 0 & 1 & 0 & 0 & 0 & 1 & 0 & -1 \end{bmatrix}.$$

In order to find $\langle R, T\rangle$- or $\langle R, S\rangle$-invariant 1-dimensional subspaces in $\mathbb{Z}_p$, it is useful to consider R, T and S as matrices over the splitting field $\mathbb{Z}_p(\zeta)$ where ζ is a solution of the polynomial $x^2 + x + 1 = 0$. The respective characteristic and minimal polynomials of R, T and S are

$$\begin{aligned}
&\Delta_R(x) = (x-1)(x-\zeta)^4(x-\zeta^2)^4, \qquad m_R(x) = (x-1)(x-\zeta)(x-\zeta^2),\\
&\Delta_T(x) = (x-1)^3(x+1)^6, \qquad m_T(x) = (x-1)(x+1),\\
&\Delta_S(x) = (x-1)^4(x+1)^5, \qquad m_S(x) = (x-1)(x+1).
\end{aligned}$$

By a straightforward calculation, we have

$$\begin{aligned}
&\mathrm{Ker}(R-I) = \langle u_1\rangle, \qquad \mathrm{Ker}(R-\zeta I) = \langle u_2, u_3, u_4, u_5\rangle, \qquad \mathrm{Ker}(R-\zeta^2 I) = \langle u_6, u_7, u_8, u_9\rangle,\\
&\mathrm{Ker}(T-I) = \langle v_1, v_2, v_3\rangle, \qquad \mathrm{Ker}(T+I) = \langle v_4, v_5, v_6, v_7, v_8, v_9\rangle,\\
&\mathrm{Ker}(S-I) = \langle w_1, w_2, w_3, w_4\rangle, \qquad \mathrm{Ker}(S+I) = \langle w_5, w_6, w_7, w_8, w_9\rangle
\end{aligned}$$

where

$$u_1 = \begin{bmatrix} 0\\0\\1\\0\\-1\\1\\1\\-1\\1 \end{bmatrix}, \quad u_2 = \begin{bmatrix} 0\\0\\0\\-\zeta-1\\1\\0\\0\\0\\0 \end{bmatrix}, \quad u_3 = \begin{bmatrix} \zeta\\-\zeta\\-1\\1\\0\\\zeta\\1\\0\\0 \end{bmatrix}, \quad u_4 = \begin{bmatrix} 0\\\zeta\\-\zeta\\0\\0\\0\\0\\0\\1 \end{bmatrix}, \quad u_5 = \begin{bmatrix} \zeta\\0\\0\\0\\0\\0\\0\\1\\0 \end{bmatrix}, \quad u_6 = \begin{bmatrix} -\zeta-1\\0\\0\\0\\0\\0\\0\\1\\0 \end{bmatrix},$$

$$u_7 = \begin{bmatrix} 0\\0\\0\\\zeta\\1\\0\\0\\0\\0 \end{bmatrix}, \quad u_8 = \begin{bmatrix} 0\\-\zeta-1\\\zeta+1\\0\\0\\0\\0\\0\\1 \end{bmatrix}, \quad u_9 = \begin{bmatrix} -\zeta-1\\\zeta+1\\-1\\1\\0\\-\zeta-1\\1\\0\\0 \end{bmatrix}, \quad v_1 = \begin{bmatrix} 0\\0\\0\\-1\\0\\1\\0\\0\\0 \end{bmatrix}, \quad v_2 = \begin{bmatrix} 0\\0\\-1\\0\\0\\0\\1\\0\\0 \end{bmatrix}, \quad v_3 = \begin{bmatrix} 0\\-1\\0\\0\\0\\0\\0\\1\\0 \end{bmatrix},$$

$$v_4 = \begin{bmatrix} 0\\0\\1\\0\\0\\0\\1\\0\\0 \end{bmatrix}, \quad v_5 = \begin{bmatrix} 0\\0\\0\\1\\0\\1\\0\\0\\0 \end{bmatrix}, \quad v_6 = \begin{bmatrix} 1\\0\\0\\0\\0\\0\\0\\0\\0 \end{bmatrix}, \quad v_7 = \begin{bmatrix} 0\\0\\0\\0\\0\\0\\0\\0\\1 \end{bmatrix}, \quad v_8 = \begin{bmatrix} 0\\0\\0\\0\\1\\0\\0\\0\\0 \end{bmatrix}, \quad v_9 = \begin{bmatrix} 0\\1\\0\\0\\0\\0\\0\\1\\0 \end{bmatrix}, \quad w_1 = \begin{bmatrix} 0\\-1\\2\\-1\\0\\0\\0\\0\\1 \end{bmatrix},$$

$$w_2 = \begin{bmatrix} -1\\1\\0\\0\\0\\0\\0\\1\\0 \end{bmatrix}, \quad w_3 = \begin{bmatrix} -1/2\\1\\-1\\1\\-1/2\\0\\1\\0\\0 \end{bmatrix}, \quad w_4 = \begin{bmatrix} 0\\0\\0\\1\\-1\\1\\0\\0\\0 \end{bmatrix}, \quad w_5 = \begin{bmatrix} 1\\0\\0\\0\\0\\0\\0\\0\\0 \end{bmatrix}, \quad w_6 = \begin{bmatrix} 0\\-1\\0\\1\\0\\0\\0\\0\\1 \end{bmatrix}, \quad w_7 = \begin{bmatrix} 0\\-1\\0\\0\\0\\0\\0\\1\\0 \end{bmatrix},$$

$$w_8 = \begin{bmatrix} 0\\0\\0\\-1\\0\\1\\0\\0\\0 \end{bmatrix}, \quad w_9 = \begin{bmatrix} 0\\0\\0\\0\\1\\0\\0\\0\\0 \end{bmatrix}.$$

Solving homogeneous linear equations over the splitting field $\mathbb{Z}_p(\zeta)$, one can see that

$$\begin{aligned}
&\mathrm{Ker}(R - I) \cap \mathrm{Ker}(T \pm I) = \mathrm{Ker}(R - I) \cap \mathrm{Ker}(S \pm I) = 0,\\
&\mathrm{Ker}(R - \zeta I) \cap \mathrm{Ker}(S \pm I) = \mathrm{Ker}(R - \zeta^2 I) \cap \mathrm{Ker}(S \pm I) = 0,\\
&\mathrm{Ker}(R - \zeta I) \cap \mathrm{Ker}(T - I) = \mathrm{Ker}(R - \zeta^2 I) \cap \mathrm{Ker}(T - I) = 0,\\
&\mathrm{Ker}(R - \zeta I) \cap \mathrm{Ker}(T + I) = \langle k_1 \rangle,\\
&\mathrm{Ker}(R - \zeta^2 I) \cap \mathrm{Ker}(T + I) = \langle k_2 \rangle
\end{aligned}$$

where

$$k_1 := \begin{bmatrix} \zeta \\ 1-\zeta \\ \zeta \\ -\zeta-1 \\ \zeta+2 \\ -\zeta-1 \\ \zeta \\ 1-\zeta \\ -2 \end{bmatrix} \quad \text{and} \quad k_2 := \begin{bmatrix} 1 \\ \zeta-1 \\ 1 \\ -\zeta-1 \\ 2\zeta+1 \\ -\zeta-1 \\ 1 \\ \zeta-1 \\ -2\zeta \end{bmatrix}.$$

Hence, there exist only two $\langle R, T \rangle$-invariant 1-dimensional subspaces $\langle k_1 \rangle$ and $\langle k_2 \rangle$. Furthermore, since $Sk_1 = \zeta k_2$, two spaces $\langle k_1 \rangle$ and $\langle k_2 \rangle$ induce isomorphic covering projections whose maximal lifting group is H. By considering the induced subgraph

$$\langle N_0(1, 0) \cup N_1(1, 0) \cup N_2(1, 0) \cup N_3(1, 0) \cup N_4(1, 0) \cup N_5(1, 0) \rangle$$

of CF_{16p}, one can see that the girth of CF_{16p} is 10. This completes the proof that any arc-transitive $\mathbb{Z}_p$-covering ($p \geq 5$) graph of F_{16} is isomorphic to the graph CF_{16p} with girth 10.

By Lemma 2.4, the graph $CF_{16\cdot 7}$ is 1-regular and isomorphic to F_{112A} because the girth of $CF_{16\cdot 7}$ is 10. Thus, one can assume $p \geq 11$. Let $\mathbf{p} : CF_{16p} \to F_{16}$ be the associated covering projection from the voltage assignment ξ and $A := \mathrm{Aut}(CF_{16p})$. Suppose to the contrary that CF_{16p} is s-regular for some $s \geq 2$. By Tutte [25,26], $s \leq 5$ and so $|A| \mid 16 \cdot p \cdot 48$. Thus $L := \mathrm{CT}(\mathbf{p})$ is a Sylow p-subgroup of A. Let B be the 1-regular subgroup of $\mathrm{Aut}(CF_{16p})$ lifted by $\langle \alpha, \beta \rangle$. Then $|B| = 16 \cdot 3 \cdot p$. The normality of L in B implies that $B \leq N_A(L)$, where $N_A(L)$ is the normalizer of L in A. Since $\widetilde{X}$ is at most 5-regular, $|A : N_A(L)| \mid 16$. By Sylow's theorem, the number of Sylow p-subgroups of A is $np + 1$ and $np + 1 = |A : N_A(L)|$. Since $p \geq 11$, we have $np + 1 = 1$. Thus L is normal in A. By Proposition 2.3, A/L is an s-regular subgroup of $\mathrm{Aut}(F_{16})$. This is impossible because otherwise s-regular subgroup $A/L (s \geq 2)$ of $\mathrm{Aut}(F_{16})$ lifts. This completes the proof of Theorem 2.6.

As continuation of this work, we have classified the cubic s-regular graphs of order $18p$ and $20p$ for every $s \geq 1$ and every prime p.

Acknowledgement

This paper was supported by Korea Research Foundation Grant (KRF-2007-313-C00011).

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